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Eigenvalue bounds for the Orr–Sommerfeld equation

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Estimates of the eigenvalues C belonging to the manifold of solutions of the Orr–Sommerfeld equation are constructed by application of elementary isoperimetric inequalities. The inequalities also lead to a considerable improvement on the estimate of (αR) regions of linear stability given by Synge.

As is well known (e.g. see Synge (1938) or Lin (1955, pp. 31–2)) the real and imaginary parts of the eigenvalue

$$C = c_r + ic_i,$$

of the Orr–Sommerfeld problem

$$(U-C)(\phi''-\alpha^2\phi)-U''\phi = -\frac{i}{\alpha R}(\phi^{\text{iv}}-2\alpha^2\phi''+\alpha^4\phi), \tag{1a}$$

$$\phi(0) = \phi(1) = \phi'(0) = \phi'(1) = 0, \tag{1b}$$

necessarily satisfy

$$c_i = \{Q-\bar{Q}-(\alpha R)^{-1}(I_2^2+2\alpha^2 I_1^2+\alpha^4 I_0^2)\}/(I_1^2+\alpha^2 I_0^2), \tag{2}$$

and

$$c_r = \left\{\int_0^1 [U|\phi'|^2+(\alpha^2 U+\tfrac{1}{2}U'')|\phi|^2]\,dy\right\}\Big/(I_1^2+\alpha^2 I_0^2), \tag{3}$$

where

$$I_2^2 = \int_0^1 |\phi''|^2 dy, \quad I_1^2 = \int_0^1 |\phi'|^2 dy,$$

$$I_0^2 = \int_0^1 |\phi|^2 dy, \quad Q = \frac{i}{2}\int_0^1 U'\phi\bar{\phi}'\,dy,$$

and α, R are real non-negative parameters. Consider functions $U(y)$ with two continuous derivatives and deduce results from (2) and (3) for elements ϕ of a complex-valued Hilbert space, $\bar{H}$, completed under the norm I_2^2 by the addition of limit points of sequences of four times continuously differentiable functions satisfying (1b).

The first result is obtained from the estimate

$$c_i \leqslant \frac{qI_0I_1-(\alpha R)^{-1}(I_2^2+2\alpha^2 I_1^2+\alpha^4 I_0^2)}{I_2^2+\alpha^2 I_0^2}, \quad q = U'_{\max} = \max_{y\in[0,1]} |U'(y)|, \tag{4a,b}$$

which has been given by Synge (1938) and follows from Schwarz's inequality in an obvious way.

THEOREM 1. *Let $C(\alpha, R)$ be any eigenvalue of $(1a, b)$. Then*

$$c_i \leqslant \frac{q}{2\alpha} - \frac{\pi^2+\alpha^2}{\alpha R}. \tag{5}$$

Moreover, no amplified disturbances ($c_i > 0$) of $(1a, b)$ exist if

$$\alpha R q < f(\alpha) \equiv \max\,[M_1, M_2], \tag{6}$$

where
$$M_1 = (4{\cdot}73)^2\pi + 2^{\frac{3}{2}}\alpha^3, \tag{6a}$$
and
$$M_2 = (4{\cdot}73)^2\pi + 2\alpha^2\pi. \tag{6b}$$

In the limit $\alpha R \to \infty$ the estimate (5) reduces to a known inviscid result of Høiland (1953). It is clear that the inviscid estimate always dominates the amplification rate in the real fluid. Of course, for finite αR, (5) is a real improvement over the inviscid estimate.

The estimate (6) greatly improves the result

$$\alpha R q < g(\alpha) \equiv \max\,[A_1, A_2, A_3], \tag{7}$$

$$A_1 = 2^{\frac{3}{2}}\alpha^2(\alpha^2+1)^{\frac{1}{2}}, \tag{7a}$$

$$A_2 = (2\alpha^2+1)^{\frac{1}{2}}(4\alpha^4+1)^{\frac{1}{2}}, \tag{7b}$$

$$A_3 = 2^{\frac{3}{2}}\alpha(1-\alpha^2+\alpha^3+\alpha^4)^{\frac{1}{2}}, \tag{7c}$$

which has been given by Synge (1938). For example, from (6) we calculate

$$45{\cdot}6 \simeq \min_\alpha \frac{f(\alpha)}{\alpha}, \quad \text{whereas (7) gives } 2{\cdot}74 \simeq \min_\alpha \frac{g(\alpha)}{\alpha}$$

(see figure 1). The estimate (6) (and (7)) gives sufficient conditions for stability to small-amplitude disturbances governed by $(1a, b)$. The application of Squires' theorem, which leads to $(1a, b)$, restricts deductions from (2) to two-dimensional disturbances periodic in the stream direction. In the equivalent non-linear problem it is the cross-stream disturbances which give the energy functional its minimum value (Joseph 1966).

I should like to make one final remark relative to (6) (or (7)) before proving the theorem. The estimates (6) and (7) share with true neutral curves the feature that upper and lower branches of these curves bend to the right. For each fixed value of R there exists a value $\alpha_{\min}$ below which there is linear stability and a value $\alpha_{\max}$ above which there is linear stability. Above $\alpha_{\max}$ the disturbance vorticity is so great that dissipation overcomes production of energy. On the other hand (6) and (7) also indicate that for every finite α there exists an $R(\alpha)$ above which linear stability cannot be deduced. This feature is not shared by true neutral curves. These exhibit a cut-off $\alpha = \alpha_c$ above which there is always linear stability, independent of R. In asymptotic theory neutral curves are formed from a truncated version of $(1a)$:

$$(U-c_r)(\phi''-\alpha^2\phi) - U''\phi = -(i/\alpha R)\,\phi^{\text{iv}}$$

and $(1b)$. The bending to the right of the upper branch of (6) (or (7)) is clearly a consequence of the dissipation integrals

$$2\alpha^2 I_1^2 + \alpha^4 I_0^2.$$

These are precisely the terms which are dropped in the asymptotics. It follows that dissipation of disturbance vorticity cannot explain the cut-off α and that this qualitative behaviour of the neutral curve, on the one hand, and (6) or (7), on the other, have nothing in common.

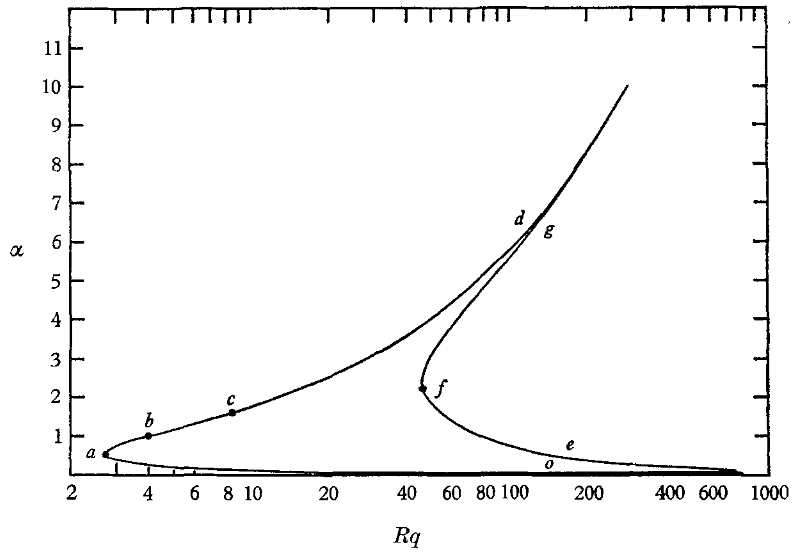

FIGURE 1. Linear stability bounds for the Orr–Sommerfeld equation. The region of certain linear stability lies to the left of the curve eg (equations (6)): ef is the graph of $M_2(\alpha)/\alpha$, fg is the graph of $M_1(\alpha)/\alpha$. The curve od is the graph of Synge's bounds (equations (7)): oa is the graph of $A_2(\alpha)/\alpha$; ab is the graph of $A_3(\alpha)/\alpha$; bc is the graph of $A_1(\alpha)/\alpha$; cd is again the graph of $A_3(\alpha)/\alpha$.

Proof of Theorem 1

In the real-valued Hilbert space H corresponding to $\bar{H}$ the following inequalities hold:

$$I_1^2 \geqslant \pi^2 I_0^2, \tag{8a}$$

$$I_2^2 \geqslant \pi^2 I_1^2, \tag{8b}$$

$$I_2^2 \geqslant (4{\cdot}73)^4 I_0^2. \tag{8c}$$

The value $(4{\cdot}73)^4$ is the smallest eigenvalue of a vibrating rod with displacement ϕ satisfying (1 *b*) (Rayleigh 1878). These inequalities also hold in $\bar{H}$. Consider

$$\phi = a + ib,$$

where $a, b \in H$. Then $$I_1^2(a) \geqslant \pi^2 I_0^2(a), \quad I_1^2(b) \geqslant \pi^2 I_0^2(b) \tag{9}$$

and by addition $$I_1^2(\phi) \geqslant \pi^2 I_0^2(\phi).$$

Equations (8 *b*) and (8 *c*) follow in the same way.

I next use the estimates (8) to establish the necessary preliminary inequalities.

$$2\alpha I_0 I_1 \leqslant I_1^2 + \alpha^2 I_0^2, \tag{10a}$$

$$I_2^2 + 2\alpha^2 I_1^2 + \alpha^4 I_0^2 = (I_2^2 + \alpha^2 I_1^2) + \alpha^2 (I_1^2 + \alpha^2 I_0^2) \geqslant (\pi^2 + \alpha^2)(I_1^2 + \alpha^2 I_0^2), \tag{10b}$$

$$\frac{I_2^2+2\alpha^2 I_1^2+\alpha^4 I_0^2}{I_0 I_1} \geqslant \begin{Bmatrix} (4\cdot 73)^2\pi+\dfrac{2\alpha^2}{I_1 I_0}\left[\left(I_1-\dfrac{\alpha}{\sqrt{2}}I_0\right)^2+\dfrac{2\alpha}{\sqrt{2}}I_0 I_1\right] \\ (4\cdot 73)^2\pi+\dfrac{2\alpha^2 I_1^2}{I_1 I_0} \end{Bmatrix} \geqslant \begin{Bmatrix} M_1 \\ M_2 \end{Bmatrix}. \tag{10c}$$

The estimate (5) follows easily from (4), (10a) and (10b). To obtain (6) consider the set of values $\{\alpha R\}$ for which the left-hand side of (4) is negative. Then apply the estimate (10c).

I turn next to estimates of the wave speed c_r. The result here is as follows.

THEOREM 2. *Let $C(\alpha, R)$ be any eigenvalue of (1 a, b). Then the following inequalities hold:*

$$(a)\quad \left.\begin{aligned} &U''_{\min} \geqslant 0,\\ &U_{\min} < c_r < U_{\max}+\frac{U''_{\max}}{2(\pi^2+\alpha^2)}.\end{aligned}\right\} \tag{11a}$$

$$(b)\quad \left.\begin{aligned} &U''_{\min} \leqslant 0 \leqslant U''_{\max},\\ &U_{\min}+\frac{U''_{\min}}{2(\pi^2+\alpha^2)} < c_r < U_{\max}+\frac{U''_{\max}}{2(\pi^2+\alpha^2)}.\end{aligned}\right\} \tag{11b}$$

$$(c)\quad \left.\begin{aligned} &U''_{\max} \leqslant 0,\\ &U_{\min}+\frac{U''_{\min}}{2(\pi^2+\alpha^2)} < c_r < U_{\max}.\end{aligned}\right\} \tag{11c}$$

Here $U_{\max}$, $U''_{\max}$, $U_{\min}$ and $U''_{\min}$ are maximum and minimum values on the range of $U(y)$ and $U''(y)$ for $y \in [0, 1]$.

The estimates (11) considerably improve Pai's (1954) extension,

$$\frac{U''_{\min}}{2\alpha^2}+U_{\min} < c_r < \frac{U''_{\max}}{2\alpha^2}+U_{\max},$$

of the wave speed bounds given by Synge (1938) for Couette and Poiseuille flow.

These estimates restrict the wave speed c_r for all parallel motions to an interval only slightly larger than the range of U. Unlike previous estimates (Synge 1938; Pai 1954) (11) shows that c_r is bounded above and below, uniformly in α.

I am unaware of calculations leading to negative wave speeds in situations covered by (11c). However, the negative wave speeds outside the range of U which are consistent with (11b) evidently do occur in Jeffery–Hamel flow in diverging channels with back flow (Eagles 1966).

Proof of Theorem 2

From (3) we find that

$$c_r = U(y_1)+\frac{\frac{1}{2}U''(y_2)}{(I_1^2/I_0^2)+\alpha^2}, \quad \infty \geqslant \frac{I_1^2}{I_0^2} \geqslant \pi^2, \tag{12}$$

where $y_1, y_2 \in (0, 1)$ are mean values. It is clear that in the situations (a, b, c) specified in the theorem we have

$$(a)\quad 0 < \frac{U''(y_2)}{(I_1^2/I_0^2)+\alpha^2} < \frac{U''_{\max}}{\pi^2+\alpha^2}, \tag{13a}$$

$$(b)\quad \frac{U''_{\min}}{\pi^2+\alpha^2} < \frac{U''(y_2)}{(I_1^2/I_0^2)+\alpha^2} < \frac{U''_{\max}}{\pi^2+\alpha^2}, \tag{13b}$$

$$(c)\quad \frac{U''_{\min}}{\pi^2+\alpha^2} < \frac{U''(y_2)}{(I_1^2/I_0^2)+\alpha^2} < 0. \tag{13c}$$

The estimates (11) now follow easily from (12) and (13).

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REFERENCES

Drazin, P. G. & Howard, L. N. 1966 Hydrodynamic stability of parallel flow of an inviscid fluid. *Adv. Appl. Mech.* **9**, 1–85.

Eagles, P. M. 1966 The stability of a family of Jeffery–Hamel solutions for divergent channel flow. *J. Fluid Mech.* **24**, 191–207.

Høiland, E. 1953 On two-dimensional perturbation of linear flow. *Geofys. Publ.* **18**, *J. Fluid Mech.* **16**, 333–42. See also Drazin & Howard (1966).

Joseph, D. D. 1966 Nonlinear stability of the Boussinesq equations by the method of energy. *Arch. Rat. Mech. Anal.* **22**, 163–84.

Lin, C. C. 1955 *The Theory of Hydrodynamic Stability.* Cambridge University Press.

Pai, S. I. 1954 On a generalization of Synge's criterion for sufficient stability of plane parallel flows. *Quart. Appl. Math.* **12**, 203–6.

Rayleigh, Lord 1878 *Theory of Sound.* 2nd ed., vol. 1, paragraph 174. New York: Dover 1945.

Synge, J. L. 1938 Hydrodynamical stability. *Semicentenn. Publ. Amer. Math. Soc.* **2**, 227–69.